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Synthesis of Stiffened Conical Shells

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The development of a method to effect the automated minimum weight design of ring and stringer stiffened shells is presented. Membrane theory is used for the shell prebuckling analysis. The buckling analysis is based upon an arbitrary shell of revolution computer program. The structural analysis includes both buckling and yielding modes of failure. The synthesis involves the coupling of an exterior penalty function with a method for the unconstrained minimization of a function comprised of a sum of squares. Results of the application of the method to the design of the Viking Aeroshell cone are presented. The least weight Viking Aeroshell appears to be an all magnesium shell with ring stiffeners of hollow circular cross section. Because the method incorporates a general shell of revolution buckling analysis, it can be readily modified and applied to the design of any axisymmetrically loaded uniformly stiffened shell of revolution for which a membrane prebuckling solution exists.

Nomenclature

a,b = dimensions of rectangular plate in θ and ξ directions
A = area of stiffener
C,K,D = shell stiffness matrices
d = diameter or web height of stiffener
E = Young's modulus
$f_i(\mathbf{x}),\phi_i(\mathbf{x})$ = constraints
G = shear modulus
H = J^TJ
i = stiffener location parameter
I = stiffener area moment
J = Jacobian matrix
$\tilde{K},\bar{K}$ = stiffness and prebuckling matrices
l = length of shell wall
$\mathbf{M},\mathbf{N}$ = moment and force resultants
n = circumferential mode number
p = uniform pressure (positive for external pressure)
P = axial load (positive for compression)
r = perpendicular distance from shell axis to meridian
R = shell radius
S = stiffener spacing
t = thickness parameter
u_ξ,u_θ,w = displacements of shell middle surface
W = weight of shell
$\mathbf{x}$ = vector of design variables
Y = yield stress
$\mathbf{Z}$ = eigenvector for general shell buckling
α = cone half angle
β,δ,η = shell loading parameters
ε = stiffener eccentricity
$\boldsymbol{\varepsilon},\boldsymbol{\kappa}$ = extensional and bending strains
λ = loading parameter
ν = Poisson's Ratio
ξ,θ = meridional and circumferential coordinates
ρ = weight density
$\boldsymbol{\sigma},\boldsymbol{\psi}$ = vectors in penalty function method
$\boldsymbol{\tau}$ = stress vector
$\phi_\xi,\phi_\theta,\phi$ = rotations of normal to shell middle surface
Φ = penalty function

Subscripts

a,b = actual and buckling value
R,S = ring and stringer quantities
t = stiffener torsional properties
W = wall quantities

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Introduction

THE last decade has seen the application of computer oriented optimization techniques to the design of structures ranging from simple structural systems such as trusses[1] to the automated preliminary design of an entire aircraft wing.[2]

Stiffened thin walled shells are widely used in the aerospace environment because they provide sufficient structural stiffness and strength at low weight. Most shell design effort to date has involved stiffened cylinders.[3-5] Cohen[6] has studied the minimum weight design of ring stiffened conical shells subjected to external pressure loading from the point of view of simultaneous failure modes, considering buckling only.

The development presented here can handle conical shells (for which the prebuckling stress and displacement state is adequately represented by the linear membrane solution) subjected to both axisymmetric pressure loading and axisymmetric axial force loading, and it includes both stiffness (buckling) and strength (yielding) failure modes (constraints). A computer program has been developed and used to design cylinders, cones, and cones with slight meridional curvature. Computer time (CPU) required for a typical stiffened cone design, considering one load condition (pressure and/or axial load), is about 5–8 min on a CDC 6600 computer.

Statement of Problem

The configuration of a typical conical shell is shown in Fig. 1. The top and bottom radii of the shell (R_1 and R_2) are prescribed, as are the cone angle (α) and the manner of support of the edges of the shell at $\xi = 0$ and $\xi = l$. Each load condition consists of a pressure loading and/or an axial loading. Three load conditions should provide sufficient flexibility.

Sections of the shell wall are shown in Fig. 2. The circumferential stiffeners are referred to as rings, and the meridional stiffeners are called stringers. The cross-sectional geometry of the stiffeners is arbitrary, except that it must be possible to specify it in terms of two free parameters which will be called design variables. All rings will have the same cross-sectional dimensions, as will all stringers. The spacings of the stiffeners are also design variables. The rings and stringers will be uniformly spaced. They can be placed either on the inside or the outside of the shell. Figure 2 shows interior circular cross section rings and exterior Z cross section stringers. The design program does not permit the stiffeners to migrate. The thickness of the thin isotropic wall of the shell is also a free parameter which is available for design purposes. Thus, there are seven design variables for this structure: the wall thickness (t_W), stringer thickness (t_S), ring thickness (t_R), stringer diameter or web depth (d_S), ring diameter or web depth (d_R), the stringer spacing at the top edge of the shell (S_S) and the ring spacing (S_R).

These seven design variables can be conveniently arrayed in a vector or column matrix $\mathbf{x}$ where $\mathbf{x}^T = (t_W, t_S, t_R, d_S, d_R, S_S, S_R)$. A particular $\mathbf{x}$ represents a particular design or point in a

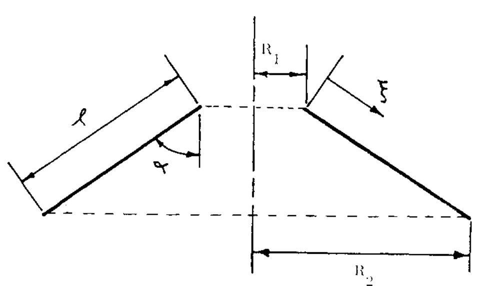

Fig. 1 Conical shell geometry.

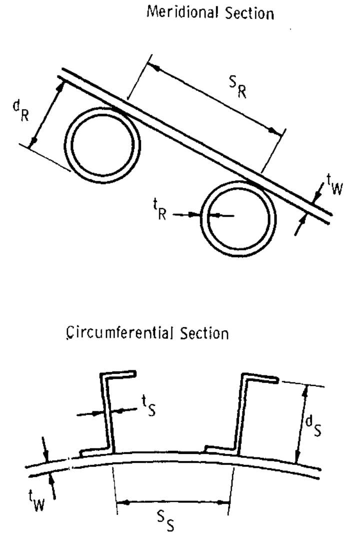

Fig. 2 Typical wall geometry.

"design space." The shell design problem is then to determine an $\mathbf{x}$ ($= \mathbf{x}^*$ say) such that the weight of the shell is minimized while the shell performs its intended function, i.e., to carry the applied load, without structural failure.

Structural Analysis

The structural analysis provides a means for predicting whether or not a particular design $\mathbf{x}$ will fail. First, the possible modes of failure for the structure must be identified, then an appropriate analytical expression or computer program must be developed to determine the behavior of a particular design relative to each failure mode.

The failure modes considered to be important for the stiffened conical shell are 1) general buckling of the entire shell, 2) buckling of the shell wall between adjacent rings and stringers (wall buckling), 3) buckling of a portion of the shell wall between adjacent rings (panel buckling), 4) local buckling in the stringers, 5) local buckling in the rings, 6) yielding in the wall, 7) yielding in the stringers, and 8) yielding in the rings. A discussion of each failure mode follows.

1. General Buckling

For general buckling, the rings and stringers are "smeared" over the surface of the shell. The resulting equivalent orthotropic shell is then analyzed by a version of the SALORS[7,8] shell analysis program. The SALORS program is based upon a finite difference formulation of Sanders' nonlinear shell equations[9] and is capable of performing stress, buckling, and vibration analyses for general shells of revolution. The version of the SALORS program incorporated into the shell design program under discussion here includes only the buckling analysis portion of the SALORS program. The buckling equations are derived from Sanders' nonlinear shell equations by separating the total rotations, displacements, force, and moment results into the sum of two parts, the first being associated with the general axisymmetric prebuckling state and the second with an infinitesimal asymmetric perturbation in the rotations, displacements, force, and moment quantities about the prebuckling state. The two sets of variables referred to above will be identified as the prebuckling variables and the buckling variables, respectively. The resulting equations are linear in the buckling variables and permit a product form of solution in which all buckling variables vary harmonically in the circumferential direction. The buckling equations are thus reduced to a set of ordinary differential equations which depend on a parameter n, the circumferential wave number. These equations are

$$\gamma N_\xi + N_\xi' + (n/r)N_{\xi\theta} - \gamma N_\theta - \omega_\xi \gamma M_\xi + \omega_\xi M_\xi' - \omega_\xi \gamma M_{\xi\theta} + \tfrac{1}{2}(n/r)(3\omega_\xi - \omega_\theta)M_{\xi\theta} - \omega_\xi \bar{N}_\xi \phi_\xi + \tfrac{1}{2}(n/r)(\bar{N}_\xi + \bar{N}_\theta)\phi + \omega_\xi \bar{\phi}_\xi N_\xi = 0 \quad (1a)$$

$$2\gamma_\xi N_{\xi\theta} + N_{\xi\theta}' - (n/r)N_\theta' + \tfrac{1}{2}[(\omega_\xi + 3\omega_\theta)\gamma - \omega_\xi']M_{\xi\theta} + \tfrac{1}{2}(3\omega_\theta - \omega_\xi)M_{\xi\theta}' - \omega_\theta(n/r)M_\theta - \omega_\theta \bar{N}_\theta \phi_\theta - \tfrac{1}{2}(\bar{N}_\xi' + \bar{N}_\theta')\phi - \tfrac{1}{2}(\bar{N}_\xi + \bar{N}_\theta)\phi' + \omega_\theta \bar{\phi}_\xi N_{\xi\theta} = 0 \quad (1b)$$

$$-\omega_\xi N_\xi - \omega_\theta N_\theta - \omega_\xi \omega_\theta M_\xi' + 2\gamma M_\xi + M_\xi'' + [\omega_\xi \omega_\theta - (n/r)^2]M_\theta - \gamma M_\theta' + 2\gamma(n/r)M_{\xi\theta} + (2n/r)M_{\xi\theta}' - (\gamma \bar{N}_\xi + \bar{N}_\xi')\phi_\xi + \bar{N}_\xi \phi_\xi' + (n/r)\bar{N}_\theta \phi_\theta (\gamma \bar{\phi}_\xi + \bar{\phi}_\xi')N_\xi + \bar{\phi}_\xi N_\xi' + (n/r)\bar{\phi}_\xi N_{\xi\theta} = 0 \quad (1c)$$

where $(\ )' \equiv d(\ )/d\xi$, $\gamma = r'/r$, $\omega_\theta = [1 - (r)^2]^{\frac{1}{2}}/r$, and $\omega_\xi = -(\gamma' + \gamma^2)/\omega_\theta$. N_ξ, N_θ, and $N_{\xi\theta}$ are the amplitudes of the harmonically varying (in the θ direction) buckling force resultants M_ξ, M_θ, $M_{\xi\theta}$ are the corresponding moment resultants, ϕ_ξ, ϕ_θ, and ϕ are the amplitudes of the harmonically varying rotations about the circumferential, meridional, and normal axes, respectively. The barred symbols are the axisymmetric prebuckling force, moment and rotation quantities, and r is the distance from the meridian to the longitudinal axis of the shell, measured perpendicular to the shell axis. For a conical shell, $r = R_1 + \xi \sin\alpha$.

The strain-displacement equations corresponding to Eqs. (1) are

$$e_\xi = u_\xi' + \omega_\xi w + \bar{\phi}_\xi(\omega_\xi u_\xi - w') \quad (2a)$$

$$e_\theta = \gamma u_\xi + (n/r)u_\theta + \omega_\theta w \quad (2b)$$

$$e_{\xi\theta} = -\tfrac{1}{2}(n/r)u_\xi - \tfrac{1}{2}\gamma u_\theta + \tfrac{1}{2}\bar{\phi}_\xi(\omega_\theta u_\theta + (n/r)w) + \tfrac{1}{2}u_\theta' \quad (2c)$$

$$k_\xi = \omega_\xi u_\xi' + \omega_\xi' u_\xi - w'' \quad (2d)$$

$$k_\theta = \gamma \omega_\xi u_\xi + (n/r)\omega_\theta u_\theta + (n/r)^2 w - \gamma w' \quad (2e)$$

$$k_{\xi\theta} = -\tfrac{1}{4}(n/r)(3\omega_\xi - \omega_\theta)u_\xi + \tfrac{1}{4}(3\omega_\theta - \omega_\xi)u_\theta' + \tfrac{1}{4}\gamma(\omega_\xi - 3\omega_\theta)u_\theta + (n/r)w' - \gamma(n/r)w \quad (2f)$$

where e_ξ, e_θ, and $e_{\xi\theta}$ are the amplitudes of the harmonically varying extensional and shearing strains, k_ξ, k_θ, and $k_{\xi\theta}$ are the amplitudes of the curvature changes, and u_ξ, u_θ, and w are the amplitudes of the meridional, circumferential, and normal, displacements. The rotations ϕ_ξ, ϕ_θ, and ϕ are given in terms of the displacements u_ξ, u_θ, and w in Sanders' paper.[9]

The constitutive relation for the elastic shell is taken in the form

$$\begin{Bmatrix} \mathbf{N} \\ \mathbf{M} \end{Bmatrix} = \begin{bmatrix} C & K \\ K & D \end{bmatrix} \begin{Bmatrix} \boldsymbol{\varepsilon} \\ \boldsymbol{\varkappa} \end{Bmatrix} \quad (3)$$

where $\mathbf{N}^T = (N_\xi, N_\theta, N_{\xi\theta})$, $\mathbf{M}^T = (M_\xi, M_\theta, M_{\xi\theta})$, $\boldsymbol{\varepsilon}^T = (e_\xi, e_\theta, e_{\xi\theta})$, $\boldsymbol{\varkappa}^T = (k_\xi, k_\theta, k_{\xi\theta})$ and C,K, and D are 3×3 matrices given in the appendix.

The shell is axisymmetric and is loaded with axisymmetric pressure p and force P. Thus, the prebuckling stress and displacement states will also be axisymmetric. The shell edges are ring supported. The load is transmitted out of the shell through the upper ring support. A linear membrane analysis is used to compute the prebuckling stress distribution. The prebuckling displacements are small and are neglected in the buckling load calculations. The prebuckling force and moment resultants are:

$$\bar{N}_\xi = \lambda\{\tfrac{1}{2}(\beta/l^2 \cos\alpha)[(R_2{}^2 - r^2)/r] - (\delta/2\pi r)/\cos\alpha\} \quad (4a)$$

$$\bar{N}_\theta = \lambda(-\beta\, r/l^2 \cos\alpha) \quad (4b)$$

In Eqs. (4), β is a dimensionless parameter proportional to the ratio of the actual pressure p to the actual axial load P as long as the latter is nonzero. Thus, if $P \neq 0$, $\beta = l^2\, p/P$, $\delta = 1$, and $\lambda = P_b$, the buckling value of P, but if $P = 0$, set $\beta = 1$, $\delta = 0$ and $\lambda = l^2 p_b$, where p_b is the buckling pressure. Also,

$$\bar{N}_{\xi\theta} = \bar{M}_\xi = \bar{M}_\theta = \bar{M}_{\xi\theta} = 0 \quad (4c)$$

and the prebuckling displacements are taken to be

$$\bar{\phi}_\xi = \bar{\phi}_\theta = 0 \quad (5)$$

The prebuckling solution satisfies over-all force equilibrium but no conditions on displacement. The buckling displacements and forces, on the other hand, must satisfy conditions which are compatible with the actual conditions which would be imposed by the end supports rings. The conditions used here are

$$\cos\alpha N_\xi - \sin\alpha N_\theta = 0, \quad u_\theta = 0$$
$$\sin\alpha u_\xi + \cos\alpha w = 0, \ M_\xi = 0 \quad (6)$$

at both top and bottom edges. These boundary conditions will be referred to as simulated ring boundary conditions.

Equations 1, with inputs from Eqs. (2–6) are solved by means of a finite difference approximation, the details of which can be found in Ref. 7. In matrix form, these finite difference equations are

$$\tilde{K}\mathbf{Z} = -\lambda | \bar{K}\mathbf{Z} \quad (7)$$

where $\tilde{K}$ is the stiffness matrix, $\bar{K}$ is the prebuckling matrix, $\mathbf{Z}$ is a vector of displacement and force quantities evaluated at each finite difference station along the meridian, and λ is the as yet undetermined multiplier of the prestress distribution. The eigenvalue problem represented by Eq. (7) is solved iteratively by the inverse power method. This method converges on the eigenvalue which is smallest in absolute value, i.e., closest to the $\lambda = 0$ axis. The sign of λ is computed by the Rayleigh Quotient formula $\lambda = -\mathbf{Z}^T\tilde{K}\mathbf{Z}/\mathbf{Z}^T\bar{K}\mathbf{Z}$ and the axis about which the eigenvalues are determined is shifted sequentially until $\mathrm{sgn}(\lambda) = \mathrm{sgn}(\lambda_a)$ where $\lambda_a = P$ if $P \neq 0$ or $l^2 p$ if $P = 0$. This shift is necessary, since if the signs of λ and λ_a are not the same, then the computed buckling load and mode shape correspond to those which would occur if the actual load were reversed. When the situation $\mathrm{sgn}(\lambda) = \mathrm{sgn}(\lambda_a)$ is obtained, λ is the buckling load, denoted λ_b. Computing $\phi_1 = \lambda_a/\lambda_b$, general buckling will not occur if $\phi_1 < 1$. Finally, it should be noted that the determination of λ_b sketched above must be repeated for each of several circumferential wave numbers n until the most critical λ_b is found.

2. Wall Buckling

For general buckling, the shell is analyzed as an equivalent orthotropic shell. This makes it important to determine at what load the wall of the actual shell will buckle. A segment of the shell wall between adjacent rings and stringers is imagined cut from the shell wall. This shell segment is then analyzed as a simply supported rectangular flat plate of dimensions (b, a) in the (ξ, θ) directions, respectively, under biaxial stress. Some comments regarding this approximation are in order. Minimum weight designs of stiffened shells generally consist of an extremely thin wall braced by many small closely spaced stiffeners. In this way, high-flexural stiffness is obtained with a minimum of structural material. It is to be expected, then, that the size of the shell segment will be small compared to the radius of curvature of the shell and that the effect of curvature on the buckling strength of the segment will be negligible. The dimension a is the average stringer spacing for the shell segment, and dimension b of the rectangular plate is the ring spacing for the shell segment.

The loads acting on the shell wall segment are derived from the prebuckling stress state, Eqs. (4), with $\lambda = \lambda_a$. The proportion of the imposed loads supported by each of the three elements which make up the complete stiffened shell, i.e., the wall, rings, and stringers, is determined from the condition of compatibility of strains. This is the requirement that the strain at any point in the actual shell be the same as the strain at the corresponding point in the equivalent orthotropic shell. Then, using Eqs. (4) with $\lambda = \lambda_a$ for the right-hand side of Eq. (3), the strains in the equivalent orthotropic shell are

$$\bar{\boldsymbol{\varepsilon}} = C^{-1}\bar{\mathbf{N}} \qquad \bar{\kappa} = 0 \tag{8}$$

For an isotropic, linearly elastic wall material, the wall stresses are

$$\boldsymbol{\tau}_W = \frac{E_W}{1 - \nu_W{}^2}\begin{bmatrix} 1 & -\nu_W & 0 \\ -\nu_W & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \bar{\boldsymbol{\varepsilon}} \tag{9}$$

where $\boldsymbol{\tau}_W{}^T = (\tau_\xi, \tau_\theta, 0)$, the subscript W refers to the shell wall. The force resultants N_1 and N_2 acting on the rectangular plate approximation to the wall segment in the ξ and θ directions, respectively, are derived from Eq. (9) by taking the average stress on each edge and dividing it by the wall thickness t_W. Then setting $\eta = N_2/N_1$, the ratio of the actual loads acting on the plate, the critical meridional load can be computed from the formula[10]

$$N_{1_b} = \min_{p,q}[\tilde{D}\pi^2(p^2/a^2 + q^2/b^2)^2/(\eta p^2/a^2 + q^2/b^2)] \tag{10}$$

where $\tilde{D}$ is given in the Appendix and p and q are the circumferential and meridional wave numbers. If $N_1 = 0$, the roles of N_1 and N_2 are interchanged. To determine if the shell will fail in wall buckling, the actual and buckling loads are compared as follows. Define ϕ_2 to be the ratio of the actual load to the buckling load. Then compute N_{1_b} from Eq. (10). If $\mathrm{sgn}(N_1) = \mathrm{sgn}(N_{1_b})$, set $\phi_2 = N_1/N_{1_b}$, but if $\mathrm{sgn}(N_1) \neq \mathrm{sgn}(N_{1_b})$ set $N_{1_b} = M\,\mathrm{sgn}(N_1)$, M a large positive number, and again compute $\phi_2 = N_1/N_{1_b}$. The shell will not fail in wall buckling if $\phi_2 < 1$.

It should be noted that the meridional variation in shell wall segment size and load makes it necessary to consider all wall segments between any two adjacent stringers for possible wall buckling.

B. Panel Buckling

For shells with no stringers or with stringers that are small compared to the rings, the possibility exists that the shell could buckle between adjacent rings, with the stringers having no or little influence on the buckle pattern. For this failure mode, the portion of the shell wall between adjacent rings is modeled as an orthotropic simply supported plate of infinite extent ($a \to \infty$) and of width b equal to the spacing of two adjacent rings. The loads on this plate are the same as those on the rectangular panel, and the buckling loads can be obtained from the formula[11]

$$N_{2_b} = (\pi^2/b^2)\{\tilde{H} + 2(D_{22})^{1/2}[D_{11} - (b^2/\pi^2)N_1]^{\frac{1}{2}}\} \tag{11}$$

with $\tilde{H} = 2(D_{12} + D_{33})$, where D_{11}, D_{12}, D_{22}, and D_{33} are as in the appendix with the ring area A_R, ring moment of inertia I_R, and ring torsional stiffness parameter I_{t_R}, equal to zero. This formula represents an extension to an orthotropic plate of the results given in Ref. 12. To determine whether or not the shell will fail in panel buckling, the actual loads must be compared to the buckling loads. Let ϕ_3 denote the ratio of the actual load to the critical load. Then if $N_1 \geq \pi^2 D_{11}/b^2$, set $N_{1b} = \pi^2 D_{11}/b^2$ and $\phi_3 = N_1/N_{1b}$. If $N_1 < \pi^2 D_{11}/b^2$, compute N_{2_b} from Eq. 11. Then, if $\mathrm{sgn}(N_2) = \mathrm{sgn}(N_{2_b})$, set $\phi_3 = N_2/N_{2_b}$, but if $\mathrm{sgn}(N_2) \neq \mathrm{sgn}(N_{2_b})$, set $N_{2_b} = M\,\mathrm{sgn}(N_2)$, M a large positive number, and compute $\phi_3 = N_2/N_{2_b}$. The shell will not fail in panel buckling if $\phi_3 < 1$. As with wall buckling, this analysis must be repeated for each panel.

4. Local stringer buckling

The actual stress in the stringers at any meridional location can be computed from strains in the equivalent orthotropic shell, as

$$\tau_S = E_S \bar{e}_\xi \tag{12}$$

where E_S is the Young's Modulus of the stringer material. The buckling load depends on the geometry of the cross-section of the stringer. If the stringers are of Z cross section, Ref. 13 gives

$$\tau_{S_b} = 5.53 E_S(t_S/d_S)^2 \tag{13}$$

This formula is based on a stringer web-flange ratio of 2.5. Defining ϕ_4 to be the ratio of the actual to buckling stress in the stringer, if $\mathrm{sgn}(-\tau_S) = \mathrm{sgn}(\tau_{S_b})$, then $\phi_4 = (\tau_S/\tau_{S_b})$ with τ_{S_b} given by Eq. (13). If $\mathrm{sgn}(-\tau_S) \neq \mathrm{sgn}(\tau_{S_b})$, then $\tau_{S_b} = M\,\mathrm{sgn}(-\tau_S)$, and again $\phi_4 = \tau_S/\tau_{S_b}$. Local stringer buckling will not occur if $\phi_4 < 1$. Since τ_S is a function of meridional location, ϕ_4 must be checked at all meridional positions.

5. Local ring buckling

The actual stress in a ring is

$$\tau_R = E_R\,\bar{e}_\theta \tag{14}$$

where E_R is the Young's Modulus of the ring. The value of τ_R will in general vary from ring to ring. For rings of hollow circular cross section, the buckling load is given by[14]

$$\tau_{R_b} = 0.4 E_R t_R/d_R \tag{15}$$

The ratio of actual to buckling stress, $\phi_5 = \tau_R/\tau_{R_b}$ is computed in the same manner as that for stringer buckling. Ring buckling will not occur if $\phi_5 < 1$.

6. Material yield in the wall

If the quantity ϕ_6 is defined to be

$$\phi_6 = (\tau_\xi{}^2 - \tau_\xi\tau_\theta + \tau_\theta{}^2)^{\frac{1}{2}}/Y_W \tag{16}$$

with Y_W denoting the uniaxial yield strength of the wall material, the Hencky-von Mises yield criterion states that yield will not occur in the wall as long as $\phi_6 < 1$.

7. Yield in the stringers

The stress in the stringers is given by Eq. (12). Denoting the uniaxial yield stress of the stringer material by Y_S, and defining ϕ_7 as

$$\phi_7 = |\tau_S|/Y_S \tag{17}$$

stringer yield will not occur as long as $\phi_7 < 1$. Note that ϕ_7 depends on meridional location.

8. Yield in the rings

The formulation for this failure mode is identical to that for stringer yield, i.e., $\phi_8 = |\tau_R|/Y_R$, where τ_R is computed from Eq. (14) and Y_R is the uniaxial yield stress of the ring material. ϕ_8 also varies from ring to ring along the meridian.

In addition to the failure modes considered previously, a design is also said to fail if the design variables do not satisfy minimum gage requirements and fabrication requirements which do not allow two stiffeners to overlap one another. For example, a typical minimum gage constraint is

$$\phi_9 = (t_R)_{\min}/t_R \tag{18}$$

where $(t_R)_{\min}$ is the minimum gage ring thickness. Similar constraints are placed on the maximum sizes of the design variables to insure that all designs investigated be in the region of validity of the analysis used. A typical fabrication constraint is the requirement that the diameter of a circular ring (or flange of a Z ring) be smaller by a certain amount than the ring spacing, as

$$\phi_{10} = (d_R + \Delta)/S_R \tag{19}$$

where Δ is the clear distance between rings. An identical expression is applicable to the stringers.

In summary, there are 24 failure modes or constraints governing the design of the stiffened conical shell: 8 behavior constraints, 7 minimum gage constraints, 7 maximum gage constraints, and 2 fabrication constraints. An acceptable design is one for which $\phi_i < 1$, $i = 1, \ldots, 24$.

Structural Design

The design variables $\mathbf{x}$ and the failure modes or constraints, $\phi_i(\mathbf{x}) < 1$, $i = 1,2, \ldots, 24$ have been discussed in previous sections. Generally there exist an infinity of designs $\mathbf{x}$ for which $\phi_i(\mathbf{x}) < 1$. The design problem is to choose the one design from among this infinity which is best in some sense. The best stiffened cone design is assumed here to be the design which minimizes the total weight of the shell W where

$$W = \pi\rho_W(R_1 + R_2)lt_W + \rho_S n_S l A_S + 2\pi A_R \rho_R n_R(R_1 + \tfrac{1}{2}l \sin\alpha + \tfrac{1}{2}i_R d_R \cos\alpha) \quad (20)$$

In this formula, $n_R = (l/S_R) + 1$ is the number of stiffening rings, $n_S = 2\pi R_1/S_S$ is the number of stringers, and i_R depends on ring location (see the appendix). Also, ρ represents weight density and A represents stiffener cross-sectional area.

The three basic ingredients for any numerical design problem have now been identified for the stiffened shell design problem namely, the design variables $\mathbf{x}$, the constraints $\phi_i(\mathbf{x})$ and the objective function $W(\mathbf{x})$. For numerical computations, it is convenient to redefine the constraints as $f_i(\mathbf{x}) = 1 - \phi_i(\mathbf{x})$, and to consider as acceptable any design for which $\phi_i(\mathbf{x}) \leq 1$, or $f_i(\mathbf{x}) \geq 0$, $i = 1, \ldots, 24$. The design problem can now be stated concisely as find $\mathbf{x} = \mathbf{x}^*$ such that $W(\mathbf{x}^*) \to \min$ while $f_i(\mathbf{x}^*) \geq 0$, $i = 1, \ldots, 24$.

Solution of the Design Problem

Two methods have been used to solve this shell design problem. The first involves the coupling of the Fiacco McCormick (FM) penalty function method with the Davidon-Fletcher Powell (DFP) method. This method is well documented,[15,16] has been widely used,[17,18] and will not be considered in detail here. The second method involves the coupling of a penalty function method due to Powell[19] with a method for unconstrained minimization developed by Marquardt.[20] An outline of some of the details of this second method follows.

The method due to Powell [which will be referred to here as the Powell Penalty Function (PPF) method] is an exterior penalty function method originally intended to solve design problems with equality constraints. In order to use this method for the present problem, which has inequality constraints [$f_i(\mathbf{x}) \geq 0$, $i = 1, \ldots, 24$], the designer must have some idea concerning which of the constraints (failure modes) will in fact control the design and these are considered to be equality constraints. Although the requirement that the designer choose the relevant failure modes in advance at first appears to be a liability of the method, it can be an asset. For instance, it is seldom that the designer does not have some idea about which constraints are most important, and a good choice of constraints will drastically reduce the computer time required for an optimal design. Also, the method is in no way restricted to the designer's initial choice for the critical constraints, and will automatically revise this choice if necessary. It is true, of course, that a poor choice of constraints will necessitate the expenditure of more computer time in arriving at an optimal design than will a good choice.

Consider then that the designer is able to make a judicious choice for the critical constraints, and let the indices of the constraints be identified as members of a set J_c. Then the PPF method will minimize the new function

$$\Phi(\mathbf{x}, \boldsymbol{\sigma}, \boldsymbol{\psi}) = W(\mathbf{x}) + \sum_{j\varepsilon J_c} \sigma_j[f_j(\mathbf{x}) + \psi_j]^2 \quad (21)$$

where the vectors $\boldsymbol{\sigma}$ and $\boldsymbol{\psi}$ are adjusted in such a way that the $\mathbf{x} = \mathbf{x}^*$ which minimizes Φ also minimizes W while satisfying the constraints $f_j(\mathbf{x}^*) = 0\, j\varepsilon J_c$. The manner in which $\boldsymbol{\sigma}$ and $\boldsymbol{\psi}$ are adjusted to accomplish this is especially simple and is treated in detail in Powell's paper.[19] The rationale behind the introduction of this function and its theoretical justification are also treated in this paper.

Once the function Φ in Eq. (21) has been minimized for a particular choice of critical constraints J_c, the next step is to determine whether or not J_c was the correct choice. There are two ways in which J_c may not have been the correct choice: 1) a constraint not in J_c has been violated, or 2) a constraint in J_c should be excluded from J_c. The first case is handled by computing all constraints after the minimum of Φ is obtained. Any constraint not included in J_c but now violated is added to the set J_c. The second case is treated by means of the Kuhn-Tucker conditions.[21,22] These conditions provide a means to identify those constraints (considered to form a subset J_c') in J_c which, if eliminated from J_c, would permit Φ (hence W) to assume still lower values while not (at least locally) violating the constraints in J_c'. This procedure is continued until no constraints are violated and J_c' is empty, at which time the design point $\mathbf{x}^*$ occupied is an (at least local) optimum design.

The one ingredient omitted thus far in this discussion of the solution to the design problem is how to minimize the function Φ. The PPF method requires the minimization of Φ with respect to $\mathbf{x}$ for a sequence of choices for $\boldsymbol{\sigma}$ and $\boldsymbol{\psi}$, with $\boldsymbol{\sigma}$ and $\boldsymbol{\psi}$ held fixed during each minimization cycle.

Note that Φ as given in Eq. (21) can be written as a sum of squares. If the number of elements of the set J_c is denoted by m, and

$$y_1 \equiv (\sigma_1)^{\frac{1}{2}}(f_1 + \psi_1),\ y_2 \equiv (\sigma_2)^{\frac{1}{2}}(f_2 + \psi_2), \ldots, y_{m+1} \equiv W^{\frac{1}{2}} \quad (22)$$

then Eq. (21) can be written in the form

$$\Phi = \mathbf{y}^T\mathbf{y} \quad (23)$$

where $\mathbf{y}$ is an $(m + 1) \times 1$ column vector. In this form, Φ is a sum of squares. A powerful method for the unconstrained minimization of a function of this type was given by Marquardt.[20] Let the Jacobian matrix be denoted by J where $J_{ij} = \partial y_i/\partial x_j$, let the gradient of Φ be denoted by $\mathbf{g}$ where $g_j = \partial\Phi/\partial x_j$; set $H = J^TJ$. The Marquardt (MARQ) method minimizes the function Φ by moving through the design space from point $\mathbf{x}_i$ to $\mathbf{x}_{i+1}$ by computing the move size and direction $\Delta\mathbf{x} = \mathbf{x}_{i+1} - \mathbf{x}_i$ from the formula

$$\Delta\mathbf{x} = -\tfrac{1}{2}(H + \Lambda I)^{-1}\mathbf{g} \quad (24)$$

where I is the identity matrix and Λ is a non-negative parameter. Marquardt's method involves a simple sub-optimization problem in which Λ is adjusted in an attempt to yield the most rapid decrease in the function Φ.

The MARQ method computes the search direction and the distance to be moved in this direction in one operation. This information is, in most cases, generated with only two evaluations of the function Φ. The DFP method, on the other hand, computes only the search direction. How far to move in this direction involves a one-dimensional search for the minimum along the line defined by the search direction, and this usually involves many evaluations of the function Φ. Note, however, that the MARQ method requires the inversion of a matrix, equal in size to the number of design variables in the problem each time the function Φ is evaluated. The DFP method requires no inversion.

It can be concluded from the foregoing remarks that the MARQ method will probably be superior to the DFP method for that class of problems (to which both methods are applicable) which involve a small number of design variables but an objective function (Φ) that requires extensive computer time to evaluate. The stiffened shell problem falls into this class.

Both the DFP and the MARQ methods require the computation of the derivatives of the objective function and the constraints. The computer programs developed for the present design problem compute the exact (as opposed to finite difference) derivatives of all quantities except the general buckling constraint. It is difficult to compute the exact derivative of the general buckling load because the matrices $\tilde{K}$ and $\bar{K}$ in Eq. (7) are nonsymmetric. A finite-difference operator is used to compute this derivative.

Results

The design scheme outlined previously has been developed into a computer program and applied to the design of stiffened cylinders, cones, and cones with slight meridional curvature. Results for a conical shell, the Viking Aeroshell, are presented here.

The Viking aeroshell has a 70° half angle. The radius at the large end (R_2) is 69 in. and that at the small end (R_1) is 32 in. The wall is constructed of a thin isotropic sheet and is stiffened by circumferential rings and meridional stringers. The load is a uniform external pressure of 3.75 psi and is removed from the shell at the small end. The boundary conditions for the general buckling analysis are given in Eqs. (6). Two materials, aluminum and magnesium are considered. These materials can be used for the shell wall, stringers, and rings in any combination. The properties of the aluminum are $E = 9.35 \times 10^6$ psi, $Y = 40{,}000$ psi, $\nu = 0.33$, and $\rho = 0.101$ lb/in^3. The magnesium properties are $E = 5.8 \times 10^6$ psi, $Y = 16{,}000$ psi, $\nu = 0.33$, and $\rho = 0.066$ lb/in^3. Minimum gage requirements are: minimum shell wall thickness = 0.016 in., minimum stiffener thickness = 0.01 in., minimum stiffener diameter or web height = 0.5 in., and minimum stiffener spacing = 1 in. Stiffeners with two types of cross-sectional geometry, hollow circular and Z, are studied. Also, because of the geometry and loading, the rings are the primary stiffeners. Hence, designs with rings but no stringers are considered. Seven designs having interior rings are summarized in Table 1, while designs 6 and 7 have exterior stringers as well. All designs except number 7 meet the minimum gage requirements. Number 7 was included to show the effect of the minimum gage requirements (compare designs 6 and 7).

From a least weight point of view, design 1 is the best. This is an all magnesium shell stiffened with circular cross section interior rings. It is critical in general buckling (f_1), panel buckling (f_2), and wall yield (f_6). Constraints on wall buckling (f_3), stringer buckling (f_4), and stringer yield (f_7), are left blank in Table 1 because they do not apply to this design. The ring buckling (f_5) and ring yield (f_8) constraints are not active because the ring wall thickness (t_R) is minimum gage constrained.

To assess the effects of a change in ring cross-sectional configuration, designs 1 and 5 can be compared. These are similar in all respects except that design 5 has Z cross section rings while design 1 has circular cross section rings. The change from circular rings to Z rings involves a small weight penalty (from 51.2 lb to 54.5 lb), but the rings are not minimum gage constrained. Thus, the use of the less efficient Z ring stiffeners allows the more effective use (i.e., since the circular rings are minimum gage constrained, they are too thick in terms of behavior response) of the material in the rings (note that for design 5, the ring buckling (f_5) constraint is active).

Designs 2, 3 and 4 are included in Table 1 to demonstrate how various combinations of materials affect the weight of the optimal shells. These three designs are circular ring stiffened and can be compared with design 1. The all aluminum design (number 2) is heaviest, with the aluminum wall, magnesium rings design (number 4) a close runner-up. Note, however, that the shells with aluminum walls (designs 2 and 4) are the only ones for which the stress in the wall (see constraint f_6) does not reach the material yield stress.

Table 2 displays the detailed behavior response information generated in conjunction with the evaluation of a design, in this case design 5 of Table 1. The column labeled "Allowable Value" contains buckling pressures, buckling stresses, and yield stresses. The column labeled "Actual Value" gives the values of the actual loads and stresses in this shell. The column labeled "Location of Critical Response" gives, if applicable, the meridional location at which the allowable and actual responses most closely approach each other. For instance, in the case of panel buckling, the panel nearest the

Table 1 Viking cone design summary

No.	Wall: Material t_w, in.	Stringers: Material & type	Stringers: Size, in. $t_s x d_s$ / S_s	Rings: Material & type	Rings: Size, in. $t_R x d_R$ / S_R	Weight (lb)	f_1 Gen. Buckling	f_2 Panel Buckling	f_3 Wall Buckling	f_4 Stringer Buckling	f_5 Ring Buckling	f_6 Wall Yield	f_7 Stringer Yield	f_8 Ring Yield
1	Mg	...	...	Mg	$0.01^b \times 0.97$	51.2	−0.002	0.001	...	...	0.499	−0.005	...	0.247
	0.048			O[a]	2.07		(6)[c]							
2	Al	...	...	Al	$0.01^b \times 0.75$	64.3	0.065	0.056	...	...	0.755	0.732	...	0.637
	0.036			O	1.75		(6)							
3	Mg	...	...	Al	0.013×0.79	56.0	0.000	0.017	...	...	0.715	0.056	...	0.569
	0.047			O	2.15		(6)							
4	Al	...	...	Mg	0.015×0.78	63.0	0.023	0.053	...	...	0.793	0.732	...	0.415
	0.037			O	1.75		(6)							
5	Mg	...	...	Mg	0.02×0.91	54.5	0.005	0.233	...	...	0.006	0.004	...	0.293
	0.047			Z	1.81		(6)							
6	Mg	Mg	$0.01^b \times 1.17$	Mg	$0.01^b \times 1.02$	53.5	0.398	−0.047	−0.018	...	0.356	−0.048	0.043	0.154
	0.027	Z[d]	1.26	C	1.07		(5)							
7	Mg	Mg	0.008×2.2	Mg	0.007×0.82	47.2	0.172	0.084	0.085	...	0.189	−0.052	0.089	0.059
	0.036	Z	2.6	O	1.3		(7)							

[a] Circular cross section stiffener.
[b] Active minimum gage constraint.
[c] Circumferential mode number.
[d] Z cross section stiffener with depth/flange ratio of 2.5, web perpendicular to shell wall.

Table 2 Behavior response information for design 5, Table 1

Type of response	Allowable value	Actual value	Location of critical response
General buckling pressure, psi	3.77(6)[a]	3.75	...
Panel buckling mer. stress, psi	266(T)[b]	174(T)	Bottom panel
cir.	14576(C)[c]	11185(C)	
Ring buckling stress, psi	11392(C)	11323(C)	Bottom ring
Wall yield mer. psi	16000	13653(T)	Top edge of shell
cir.,	16000	3907(C)	
Ring yield, psi	16000	11323(C)	Bottom ring

[a] Circumferential mode number.
[b] Tensile stress.
[c] Compressive stress.

large end (bottom) of the Viking shell will be the first to fail in this mode.

Conclusion

The development of a stiffened shell design program and the results of its application to the design of a typical stiffened conical shell, the Viking Aeroshell, have been presented. The design program incorporates the SALORS program for the analysis of a general shell of revolution. Because of this, the design program is not restricted to conical shells, but can be used to design any axisymmetrically loaded uniformly stiffened shell of revolution for which a membrane solution exists. Only the shell geometry, as defined by the meridional position r, and the shell prebuckling stress distribution, as defined by Eqs. (4), must be modified.

Appendix

$$\begin{bmatrix} \tilde{C}(1+\mu_S R_1/r) & \nu_W \tilde{C} & 0 \\ & \tilde{C}(1+\mu_R) & 0 \\ & \text{Symmetric} & E_W t_W/(1+\nu_W) \end{bmatrix} = C$$

$$\begin{bmatrix} E_S A_S \varepsilon_S R_1/S_S r & 0 & 0 \\ & E_R A_R \varepsilon_R/S_R & 0 \\ & \text{Symmetric} & 0 \end{bmatrix} = K$$

$$\begin{bmatrix} \tilde{D} + E_S I_{0_S} R_1/S_S r & \nu_W \tilde{D} & 0 \\ & \tilde{D} + E_R I_{0_R}/S_R & 0 \\ & \text{Symmetric} & D_{33} \end{bmatrix} = D$$

$$\tilde{C} = E_W t_W/(1-\nu_W{}^2);\ \tilde{D} = E_W t_W/12(1-\nu_W{}^2)$$

$$D_{33} = D(1-\nu_W) + \tfrac{1}{2}(G_S I_{tS})/(S_S r) + \tfrac{1}{2}(G_R I_{tR})/S_R$$

$$\mu = (1-\nu_W{}^2)EA/(E_W t_W S);\ \varepsilon = \tfrac{1}{2} i(t_W + t + d)$$

$$i = (-1,0,+1) \text{ for (interior, no, exterior) stiffeners}$$

$$I_0 = I + A\varepsilon^2$$

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